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Differential Geodesy of the Eotvos Torsion Balance

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The differential geodesy of the Eotvos torsion balance is examined by using the fourth and fifth fundamental tensors of the geopotential surfaces of a rotating Earth. It is shown that both these tensors involve curvature differences which are measurable by the Eotvos torsion balance. This suggests that the inability to determine the individual curvatures is not only a physical limitation, but also a geometric one. The fundamental tensors are expressed using the Hotine 2-leg formalism and their components are explicitly exhibited for the case of a spheroid.

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CONTENTS

1. Introduction

Although torsion balance measurements are obsolete in gravity prospecting, the Eötvös torsion balance remains a useful conceptual device in theoretical geodesy. Indeed it may be employed to measure the curvature differences and gravity gradients which are essentially the components of the Marussi tensor of geopotential gravity gradients.

Following Hotine (1969) let N denote the geopotential function for the Earth which is assumed to be rotating with a constant angular velocity $\tilde{\omega}$. Points having a common value of N, say a constant C, are said to lie on a geopotential N-surface, $N(x^1, x^2, x^3) = C$, where x^r are rectilinear Cartesian coordinates in Euclidean 3-space. The gradient of N, denoted by N_r, satisfies the *basic gradient equation*

$$N_r = n\nu_r \tag{1.1}$$

where n is the magnitude of the acceleration due to gravity, and ν_r is an upward pointing unit normal to the N-surface. Thus, although N_r is the gravity vector of the Earth, by Hotine's convention in order to give it a downward pointing direction it is necessary to take $n = -g$. We will simply retain n in our discussion. The gravity gradients are then given by the components of the *Marussi tensor*, i.e.

$$N_{rs} := N_{r,s} = n_s\nu_r + n\nu_{r,s} \tag{1.2}$$

where the comma denotes covariant differentiation with respect to the flat space metric of Euclidean 3-space. The external geopotential field N of the Earth then satisfies the field equation

$$\Delta N = -2\tilde{\omega}^2, \tag{1.3}$$

where Δ is the flat 3-dimensional Laplacian operator, and this together with the Bruns equation

$$(\log n)_r\nu^r = -\nu^r_{,r} - 2\tilde{\omega}^2/n \tag{1.4}$$

reduces the number of independent components of the (symmetric) Marussi tensor from six to five. Thus, as Marussi (1949) showed, the metrical properties of the Earth's geopotential field are completely described by five functions which Hotine (1969) denoted by $k_1, k_2, t_1, \gamma_1, \gamma_2$ and called the *curvature parameters*. An explicit identification of these curvature parameters will be given in Section 3.

An immediate question arises as to how in practice these curvature parameters can be measured. Both Marussi and Hotine suggested that this could be done by employing the Eötvös torsion balance and the latter worked out the corresponding theory in Hotine (1957, pages 9-10) and Hotine (1969, pages 150-151). In the latter reference he commented that

> "Measurement in several azimuths will accordingly determine $k_1 - k_2$, t_1, γ_1, γ_2 and some instrument constants, but will not separate k_1 and k_2. To do this, we need an additional form of measurement."

Hotine then discussed possibilities for more refined measurements and gave several references. The general theory was analyzed exhaustively in Mueller (1960), and Mueller (1963). The general conclusion is that the inability to distinguish, i.e. measure, k_1 and k_2 separately is a *physical limitation* imposed by the use of the Eötvös torsion balance.

In the present paper we will show by employing some tensorial methods recently developed by the author in Zund (1988a, 1988b) that the *average curvature differences* $\pm(k_2 - k_1)/2$ occur in expressions for the fourth and fifth fundamental tensors. This suggests that the above limitation may have a *geometric origin* since these tensors naturally arise in the differential geometry of the N-surfaces. In Section 2 we briefly review the relevant material from our papers which is required in this study. These results are reformulated in the 2-leg representation in Section 3, and finally in an

Appendix we exhibit an explicit 2-leg representation when the N-surfaces are spheroids.

In our discussion, unless specified to the contrary, our notation and terminology follows that employed in Hotine (1969), although this is slightly different from the purely mathematical presentation in Zund (1988a, 1988b). Special note should also be made of the elegant expose of the differential geometry of the gravity field recently given by E.W. Grafarend (1986) in this journal.

2. The Fourth and Fifth Fundamental Tensors and Invariant Curvatures.

It is well known that the intrinsic and extrinsic differential geometry of surfaces which are isometrically imbedded in Euclidean 3-space can be described by *three fundamental tensors*: $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$; and two invariant curvatures: the *Gauss (total) curvature* K and the *Germain (mean) curvature* H. However, one can also introduce a fourth and fifth fundamental tensor and two additional invariant curvatures. These quantities are implicit in the work of the Soviet school of tensor analysis created by V.F. Kagan (1869-1953). Unfortunately, much of this work is little known or available in the West, but recently in Zund (1988a, 1988b) we have presented a unified exposition and generalization of some of this work. We now review those results which are useful in the differential geodesy of the Eötvös torsion balance.

We begin with the *Euler tensor*

$$h_{\alpha\beta} := \epsilon^{\rho\sigma} a_{\alpha\rho} b_{\beta\sigma} \tag{2.1}$$

which occurs in the equation

$$h_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = 0 \tag{2.2}$$

which defines the principal directions λ^{α} on a surface, see McConnell (1931, page 212). The Euler tensor (2.1) has no privileged symmetry, although only its symmetric part contributes to defining the principal directions. If this tensor is decomposed into its symmetric and skew-symmetric parts, then we obtain

$$h_{\alpha\beta} = d_{\alpha\beta} + H\epsilon_{\alpha\beta}$$

where $d_{\alpha\beta} = d_{\beta\alpha}$ is the *fourth fundamental tensor*. It was employed by S.G. Gasparyan (1961) in a beautiful memoir on the bending of surfaces which leaves the Gaussian curvature invariant, however, we believe it was known many years earlier by geometers of the Moscow school. Hence, it is probably appropriate to call it the *Kagan tensor*. One can also formally introduce a *fifth fundamental tensor* $f_{\alpha\beta}$ which can be expressed in the form

$$f_{\alpha\beta} = Ha_{\alpha\beta} - b_{\alpha\beta} \ . \tag{2.4}$$

Note that $f_{\alpha\beta}$ is obviously symmetric, and by virtue of its interpretation in differential geodesy given in Section 3 we call it the *Eötvös tensor*. The detailed geometric and algebraic properties of these tensors are given in Zund (1988a, 1988b).

The two additional invariant curvatures are defined by

$$E: = H^2 - K \ , \tag{2.5}$$

$$F: = 2H^2 - K \ , \tag{2.6}$$

and are called the *Euler* and *Monge curvatures* respectively. Up to a numerical factor of one half, F is the Casorati curvature of Marussi (1951; 1985 page 19). Each of the invariant curvatures can be expressed tensorially, i.e.

$$2H = \epsilon^{\alpha\rho}\epsilon^{\beta\sigma}a_{\alpha\beta}b_{\rho\sigma} \ , \tag{2.7}$$

$$2K = \epsilon^{\alpha\rho}\epsilon^{\beta\sigma} b_{\alpha\beta} b_{\rho\sigma} , \tag{2.8}$$

$$2E = a^{\alpha\rho} a^{\beta\sigma} d_{\alpha\beta} d_{\rho\sigma} , \tag{2.9}$$

$$2F = a^{\alpha\rho} a^{\beta\sigma} b_{\alpha\beta} b_{\rho\sigma} . \tag{2.10}$$

An equivalent expression for (2.7) is $2H = a^{\alpha\beta} b_{\alpha\beta}$ which shows that the Eötvös tensor is traceless.

3. Hotine's 2-Leg Representation.

Following Grafarend (1986) we call a pair of orthogonal unit surface vectors a *2-leg*. In accord with the notation of Chapters 6 and 7 of Hotine (1969), we denote the contravariant components of the 2-leg by $\{\ell^\alpha, j^\alpha\}$ with $\ell^r := x^r_\alpha \ell^\alpha$, $j^r := x^r_\alpha j^\alpha$ being the corresponding space vectors. In the latter expressions, $x^r_\alpha := \partial x^r / \partial x^\alpha$ where the x^r are rectangular Cartesian coordinates in Euclidean 3-space, and x^α denote curvilinear coordinates, i.e. parameters, on the N-surfaces. This 2-leg may be extended to a 3-leg of space vectors $\{\ell^r, j^r, \nu^r\}$ by including the unit normal ν^r to the N-surfaces. Then the 2-legs $\{\ell^r, \nu^r\}$, $\{j^r, \nu^r\}$ define planes which are said to be *sections* of the N-surfaces.

As a consequence of the orthogonality conditions of the 2-leg, the Levi-Civita permutation tensor has the 2-leg representation

$$\epsilon_{\alpha\beta} = \ell_\alpha j_\beta - j_\alpha \ell_\beta . \tag{3.1}$$

The fundamental surface tensors are then readily found to have the representations:

$$a_{\alpha\beta} = \ell_\alpha \ell_\beta + j_\alpha j_\beta , \tag{3.2}$$

$$b_{\alpha\beta} = k_1 \ell_\alpha \ell_\beta + t_1 (\ell_\alpha j_\beta + j_\alpha \ell_\beta) + k_2 j_\alpha j_\beta , \tag{3.3}$$

$$c_{\alpha\beta} = k_1^2 \ell_\alpha \ell_\beta + (k_1 + k_2) t_1 (\ell_\alpha j_\beta + j_\alpha \ell_\beta) + k_2^2 j_\alpha j_\beta , \qquad (3.4)$$

where k_1, k_2 are the *normal curvatures*, and t_1, $t_2 = -t_1$ are the *geodesic torsions* of the N-surface in the directions ℓ^α, j^α. The remaining two curvature parameters γ_1, γ_2 are related to the curvature of the normal n^r in the directions ℓ^r, j^r and do not depend on the fundamental tensors.

The corresponding expressions for the invariant curvatures of the N-surfaces are given by

$$H = (k_1 + k_2)/2 , \qquad (3.5)$$

$$K = k_1 k_2 - t_1^2 , \qquad (3.6)$$

$$E = (k_1 - k_2)^2/4 + t_1^2 , \qquad (3.7)$$

$$F = (k_1^2 + k_2^2)/2 + t_1^2 . \qquad (3.8)$$

The 2-leg representations for the Kagan and Eötvös tensors are then found to be

$$d_{\alpha\beta} = t_1 \ell_\alpha \ell_\beta + \left(\frac{k_2 - k_1}{2}\right)(\ell_\alpha j_\beta + j_\alpha \ell_\beta) + t_1 j_\alpha j_\beta , \qquad (3.9)$$

$$f_{\alpha\beta} = \left(\frac{k_2 - k_1}{2}\right)(\ell_\alpha \ell_\beta - j_\alpha j_\beta) - t_1 (\ell_\alpha j_\beta + j_\alpha \ell_\beta) , \qquad (3.10)$$

and hence the Euler tensor has the representation

$$h_{\alpha\beta} = t_1 (\ell_\alpha \ell_\beta + j_\alpha j_\beta) + k_2 \ell_\alpha j_\beta - k_1 j_\alpha \ell_\beta . \qquad (3.11)$$

Note that these representations show that in general neither $d_{\alpha\beta}$ nor $h_{\alpha\beta}$ are traceless, viz both have the trace $2t_1$, while as mentioned previously $f_{\alpha\beta}$ is always traceless.

Equations (3.9), (3.10) indicate the first evidence of the relationship between the curvature differences and the Kagan and Eötvös tensors. This appearance is of course due to our choice of the 2-leg, but aside from the orthogonality requirement the 2-leg $\{\rho_\alpha, j_\alpha\}$ is quite general. Indeed, if this requirement were relaxed, and the unit vectors ρ^α, j^α were inclined to each other at an angle $0 < \theta < \pi$, then $a_{\alpha\beta}\rho^\alpha j^\beta = \cos\theta$, $\epsilon_{\alpha\beta}\rho^\alpha j^\beta = \sin\theta$, and our previous expressions would contain additional trigonometric functions but no new curvature parameters would occur. Thus the relationship between the curvature differences and these tensors is not essentially a consequence of a specialization of the 2-leg.

However, by a further natural specialization of the 2-leg, this representation can be put in a more transparent and striking form. This additional specialization involves choosing a 2-leg in which the normal curvatures k_1, k_2 assume their maximum and minimum values. Such a 2-leg is called a *principal* 2-leg, the corresponding vectors are *principal directions*, and the sections of the N-surfaces *principal sections*. Following Hotine (1969, page 42) these vectors are denoted by u^α, v^α and relative to this 2-leg, k_1, k_2 reduce to the *principal curvatures* κ_1, κ_2, and $t_1 = 0$. Algebraically this specialization makes u^α, v^α *eigenvectors* of the fundamental tensors $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$, $d_{\alpha\beta}$, and $f_{\alpha\beta}$. The principal 2-leg representations of these tensors are thus:

$$a_{\alpha\beta} = u_\alpha u_\beta + v_\alpha v_\beta \, , \tag{3.12}$$

$$b_{\alpha\beta} = \kappa_1 u_\alpha u_\beta + \kappa_2 v_\alpha v_\beta \, , \tag{3.13}$$

$$c_{\alpha\beta} = \kappa_1^2 u_\alpha u_\beta + \kappa_2^2 v_\alpha v_\beta \, , \tag{3.14}$$

$$d_{\alpha\beta} = \left(\frac{\kappa_2 - \kappa_1}{2}\right)(u_\alpha v_\beta + v_\alpha u_\beta) \, , \tag{3.15}$$

$$f_{\alpha\beta} = \left(\frac{\kappa_2 - \kappa_1}{2}\right) u_\alpha u_\beta + \left(\frac{\kappa_1 - \kappa_2}{2}\right) v_\alpha v_\beta \, , \tag{3.16}$$

and this representation explicitly exhibits the *eigenvalues* of these tensors in terms of the principal curvatures. Of course, this *eigenvector-eigenvalue property* does not hold for either the Levi-Civita tensor

$$\epsilon_{\alpha\beta} = u_\alpha v_\beta - v_\alpha u_\beta , \tag{3.17}$$

or the Euler tensor

$$h_{\alpha\beta} = \kappa_2 u_\alpha v_\beta - \kappa_1 v_\alpha u_\beta . \tag{3.18}$$

Higher order powers of the principal curvatures may occur, for example in

$$2Hc_{\alpha\beta} - Kb_{\alpha\beta} = \kappa_1^3 u_\alpha u_\beta + \kappa_2^3 v_\alpha v_\beta ,$$

but the analysis given in Zund (1988b) suggests that such combinations of the fundamental tensors have no particular geometric significance.

Although the Kagan tensor (3.15) involves a curvature difference, the Eötvös tensor (3.16) is more interesting since in terms of its principal 2-leg representation it exhibits *both* curvature differences. Indeed, this is why we refer to this tensor as the Eötvös tensor. It should be noted that with characteristic modesty Marussi (1931, 1985) always referred to N_{rs}, i.e. our (1.2), as the Eötvös tensor, but most mathematical geodesists have followed Hotine's custom of calling the latter the Marussi tensor. Note that since $\kappa_1 = \kappa_2$ for a sphere, the Kagan and Eötvös tensors in differential geodesy essentially measure the deviation of the spheroidal N-surfaces from perfect spherical form. The appearance of the curvature differences in these tensors is surprising since both tensors have a purely geometric origin. It also suggests that Hotine's observation quoted in Section 1 is not merely a physical limitation imposed by employing torsion balance measurements, but a geometric limitation arising from the geometry of the N-surfaces. An additional reason for this can be found in Hotine (1969, p. 44) where it is shown that the principal 2-leg representation of the Codazzi-Mainardi equations leads to the equations

$$(\kappa_1 - \kappa_2)\sigma = \kappa_{1,\alpha} v^\alpha$$

$$(\kappa_1 - \kappa_2)\sigma^* = \kappa_{2,\alpha} u^\alpha$$

where σ, σ^* are the *geodesic curvatures* of the principal directions u^α, v^α respectively.

An excellent classical discussion, without using tensors, is given in Slotnick (1932), and it is reviewed with reference to geophysical prospecting in Dobrin (1960, pp. 180-183).

We believe that the above discussion indicates not only a connection between the torsion balance and differential geometry, but also the advantages of the 2-leg approach to differential geodesy. This approach was implicit in the work of Marussi and Hotine, however, their exposition of it failed to emphasize that it is a *coordinate-free method* and that it is conceptually distinct from the classical coordinate approach to tensors. In the 2-leg approach, as we will illustrate in the following Appendix, the rôle of coordinates is a secondary one and the resulting coordinates have a natural interpretation. In effect one makes a physical, or geometrical, choice of a 2-leg, which involves measurable quantities and then introduces a coordinate system based on the 2-leg. This is of course classically the notion of *physical components* of vectors/tensors as contained in the appendix of McConnell (1931), however, only the modern approach to tensors makes this procedure natural. Both Marussi and Hotine realized the advantages of the 2-leg formalism, but were unable to separate it from their primary goal of finding *intrinsic* coordinate systems.

4. Appendix. Explicit Calculations on the Spheroid.

We now exhibit the components of the fundamental tensor, the Euler tensor, and the principal 2-legs when the N-surfaces are spheroids, i.e., ellipsoids of revolution about the x^3-axis having semi-axes $a > b$. Following Hotine (1969), we take the curvilinear surface coordinates (parameters) to be $x^\alpha = (\omega, \phi)$ where ω is the longitude and ϕ the latitude $0 \leq \omega < 2\pi$, $-\pi/2 \leq \phi \leq \pi/2$. Then the first fundamental form becomes

$$ds^2 = \rho_1^2 \cos^2\phi \, d\omega^2 + \rho_2^2 d\phi^2 \tag{A.1}$$

where $\rho_1 := 1/\kappa_1$, $\rho_2 := 1/\kappa_2$, are the *principal radii of curvature* where

$$\kappa_1 = -(1 - e^2 \sin^2\phi)^{1/2}/a \ , \tag{A.2}$$

$$\kappa_2 = -(1 - e^2 \sin^2\phi)^{3/2}/a(1 - e^2) \ , \tag{A.3}$$

and e is the *eccentricity*

$$e^2 := (a^2 - b^2)/a^2 \ . \tag{A.4}$$

In the remainder of our discussion it will be convenient to employ κ_1, κ_2, or ρ_1, ρ_2 in place of the above explicit values. Then the components of the first fundamental tensor are given by

$$\| a_{\alpha\beta} \| = \left\| \begin{matrix} \rho_1^2 \cos^2\phi \ , & 0 \\ 0 \ , & \rho_2^2 \end{matrix} \right\| \tag{A.5}$$

and by (3.12) the components of the principal 2-leg are

$$u_1 = -\rho_1 \cos\phi \ , \ u_2 = 0 \ ; \tag{A.6}$$

$$v_1 = 0 \ , \ v_2 = -\rho_2 \ ; \tag{A.7}$$

and

$$u^1 = -\kappa_1 \sec\phi \ , \ u^2 = 0 \ ; \tag{A.8}$$

$$v^1 = 0 \;,\; v^2 = -\kappa_2 \;. \tag{A.9}$$

The components of the remaining fundamental tensors are then

$$\|b_{\alpha\beta}\| = \begin{Vmatrix} \kappa_1\rho_1^2\cos^2\emptyset \;, & 0 \\ 0 \;, & \kappa_2\rho_2^2 \end{Vmatrix} \;; \tag{A.10}$$

$$\|c_{\alpha\beta}\| = \begin{Vmatrix} \kappa_1^2\rho_1^2\cos^2\emptyset \;, & 0 \\ 0 \;, & \kappa_2^2\rho_2^2 \end{Vmatrix} \;; \tag{A.11}$$

$$\|d_{\alpha\beta}\| = \begin{Vmatrix} 0 \;, & \left(\frac{\kappa_2-\kappa_1}{2}\right)\rho_1\rho_2\cos\emptyset \\ \left(\frac{\kappa_2-\kappa_1}{2}\right)\rho_1\rho_2\cos\emptyset \;, & 0 \end{Vmatrix} \;; \tag{A.12}$$

$$\|f_{\alpha\beta}\| = \begin{Vmatrix} \left(\frac{\kappa_2-\kappa_1}{2}\right)\rho_1^2\cos^2\emptyset \;, & 0 \\ 0 \;, & \left(\frac{\kappa_1-\kappa_2}{2}\right)\rho_2^2\cos^2\emptyset \end{Vmatrix} \;; \tag{A.13}$$

and the Euler tensor has components

$$\|h_{\alpha\beta}\| = \begin{Vmatrix} 0 & \rho_1\cos\emptyset \\ -\rho_2\cos\emptyset \;, & 0 \end{Vmatrix} \;. \tag{A.14}$$

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